Representation Theory I

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Contents

Summary of lectures

Orga 0.0.1. There is a Sciebo share for this course. Usually, information regarding the lecture can be found on the webpage, but exercise sheets will also be uploaded on Sciebo.

Lecture 1
We 09 Apr 2025

Tutorials will be Wednesday 12pm and Friday 2pm, starting next week.

To be admitted to the exam, you need 50% of marks on the homework sheets.

0 Representation theory of Lie algebras

Let us start by motivating why we want to study lie algebras:

- They are infinitesimal versions of linear algebra.
- Versions of Lie groups / algebraic groups.
- We will see many techniques which are non-standard in representation theory
- We want to "study symmetries" (e.g. related to physics and chemistry).

Given a lie group / algebraic group G, e.g.

- $G = \mathrm{GL}_n(\mathbb{R})$
- $G = \mathrm{SL}_n(\mathbb{R})$
- $G = \mathrm{GL}_n(\mathbb{C})$
- $G = \mathrm{SL}_n(\mathbb{C})$
- $G = \{z \in \mathbb{C} \mid \overline{z} = 1\}$

one can consider the tangent space $\mathfrak{g} := T_e G$ at the identity element $e \in \mathfrak{g}$. This will be a lie algebra.

Idea. *Say that $G \subseteq M_{n\times n}(\mathbb{C})$ is a group and $X \in M_{n\times n}(\mathbb{C})$. Then*

$$e^X := \sum_{n=0}^{\infty} \frac{X^n}{n!} \in M_{n\times n}(\mathbb{C}), \qquad \text{where by convention } X^0 := \text{identity matrix}$$

converges in the Hilbert-Schmidt norm defined by

$$\|X\| := \sqrt{\sum_{i,j} |X_{i,j}|^2}.$$

What is clear is that e^X is invertible with inverse e^{-X}, since e^0 is the identity matrix.

In this case,

$$\mathfrak{g} := T_e G = \left\{ X \in M_{n\times n}(\mathbb{C}) \;\middle|\; e^{tX} \in G \ \forall t \in \mathbb{R} \right\}$$

will be the lie algebra associated to G.

Fact 0.0.2. For $X, Y \in \mathfrak{g}$, we can consider the commutator of matrices $[X, Y] = XY - YX \in \mathfrak{g}$.

What we want is to formalize this data in the corresponding lie group. We will also want to study representation theory of lie algebras.

1 Basic definitions

Fix a field k.

Definition 1.1 (Lie algebra)**.** A **lie algebra** (over k) is a vector space $\mathfrak{g}$ together with a bilinear map $[-,-]\colon \mathfrak{g} \times \mathfrak{g} \to \mathfrak{g}$, called the **lie bracket**, satisfying

(L1) $[x, x] = 0$ for all $x \in \mathfrak{g}$. (**antisymmetry**)

(L2) $[x, [y, z]] + [y, [z, x]] + [z, [x, y]] = 0 \ \forall x,\ y,\ z \in \mathfrak{g}$. (**Jacobi identity**)

Remark 1.1.3. From (L1), we get that $[x, y] = -[y, x]$ for all $x, y \in \mathfrak{g}$. Indeed,

$$0 = [x + y, x + y] = \underbrace{[x, x]}_{0} + [x, y] + [y, x] + \underbrace{[y, y]}_{0} = [x, y] + [y, x].$$

Hence $[x, y] = -[y, x]$, which is what is usually referred to as antisymmetry. From $[x, y] = -[y, x]$, one can also deduce $2[x, x] = 0$, which is equivalent to our definition when $\operatorname{char} k \neq 2$.

Definition 1.2 (Morphism of lie algebras)**.** Let $\mathfrak{g}_1$ and $\mathfrak{g}_2$ be lie algebras. A **morphism of lie algebras** $\varphi\colon \mathfrak{g}_1 \to \mathfrak{g}_2$ is a linear map such that $\varphi([x, y]) = [\varphi(x), \varphi(y)]$ for all $x, y \in \mathfrak{g}_1$.

Lie algebras together with morphisms of lie algebras give rise to the category of lie algebras.

Example 1.3. (0) Let $\mathfrak{g} = V$ be any vector space. This can be turned into a lie algebra by setting $[v, w] := 0$ for $v, w \in V$. This special case is called an **abelian lie algebra** (i.e. $[-,-] \equiv 0$).

Note that the category of abelian lie algebras is canonically equivalent to the category of vector spaces.

(1) Let A be a k-algebra. Then A is a lie algebra via $[a, b] := ab - ba$ for $a, b \in A$, i.e. the lie bracket is the commutator in A. The Jacobi

identity is the analogue of the associativity of multiplication in A:

$$\begin{aligned} & [a,[b,c]]+[b,[c,a]]+[c,[a,b]] \\ &= [a, bc-cb]+[b, ca-ac]+[c, ab-ba] \\ &= a(bc)-a(cb)-(bc)a+(cb)a+b(ca)-b(ac)-(ca)b+(ac)b \\ &\quad + c(ab)-c(ba)-(ab)c+(ba)c \\ &= 0 \end{aligned}$$

An important special case is $A = M_{n\times n}(k)$ or $A = \mathrm{End}_k(V)$ for a vector space V. Using this construction, we get $\mathfrak{g} = \mathfrak{gl}_n(k)$ and $\mathfrak{g} = \mathfrak{gl}(V)$, called **general linear lie algebras**.

(2) Let A be a k-algebra, then we can consider the **derivations**

$$\begin{aligned} \mathrm{Der}_k(A) &:= \{\text{derivations of } A\} \\ &:= \{d\colon A \to A \mid d \text{ is } k\text{-linear}, d(ab) = d(a)b + ad(b)\ \forall a,b \in A\}. \end{aligned}$$

This is a lie algebra via $[d, d'] := d \circ d' - d' \circ d$.

Note that

$$\begin{aligned} (dd')(ab) &= d(d'(a)b + ad'(b)) \\ &= (dd'(a))b + d'(a)d(b) + d(a)d'(b) + a((dd')(b)) \end{aligned}$$

and thus by applying this formula twice

$$\begin{aligned} [d,d'](ab) &= (dd'(a))b + d'(a)d(b) + d(a)d'(b) + a(dd'(b)) \\ &\quad - \left[(d'd(a))b + d(a)d'(b) + d'(a)d(b) + a(d'd(b))\right] \\ &= (dd'(a))b - (d'd(a))b + a(dd'(b)) - a(d'd(b)) \\ &= ([d,d'](a))b + a([d,d'](b)). \end{aligned}$$

Hence we also have $[d, d'] \in \mathrm{Der}_k(A)$. The Jacobi identity follows from associativity of composition analogue to the computation in (1) .

(3) **Classical lie algebras**, i.e. lie algebras given by matrices with the lie bracket $[-,-]$ being the commutator. For example,

$$\mathfrak{sl}_n(k) := \{A \in M_{n\times n}(k) \mid \mathrm{Tr}(A) = 0\}.$$

From $\mathrm{Tr}(AB) = \mathrm{Tr}(BA)$ and linearity of trace, we immediately get that $[A, B] \in \mathfrak{sl}_n(k)$ for any two matrices. We will denote this class of lie algebras by A_n as well, suppressing k.

The **special linear lie algebra** B_n. Let $X = k^{2n+1}$ and pick a symmetric nondegenerate bilinear form $\beta = (-,-)$ on V.

Then consider

$$\mathfrak{so}_{2n+1}^{\beta} := \{A \in M_{2n+1,2n+1}(k) \mid (Av, w) + (v, Aw) = 0\}.$$

Note that for $A, B \in \mathfrak{so}_{2n+1}(k)$, we have

$$\begin{aligned}([A,B]v,w) + (v,[A,B]w) &= (ABv,w) - (BAv,w) + (v,ABw) - (v,BAw) \\ &= -(Bv,Aw) + (Av,Bw) - (Av,Bw) + (Bv,Aw) \\ &= 0\end{aligned}$$

and hence $[A, B] \in \mathfrak{so}_{2n+1}(k)$

Similarly, we can construct the lie algebras C_n. Let $V = k^{2n}$ and pick a skewsymmetric / symplectic non-degenerate bilinear form β and consider

$$\mathfrak{sp}_{2n}^{\beta}(k) := \{A \in M_{2n,2n}(k) \mid (Av, w) + (v, Aw) = 0\ v, w \in V\}.$$

This is a lie algebra and called the **symplectic lie algebra**.

D_n is the same as B_n except of $V = k^{2n}$, yielding the **(special) orthogonal lie algebra** $\mathfrak{so}_{2n}^{\beta}(k)$.

Example 1.4. Concretely, the lie algebra $\mathfrak{sl}_2(\mathbb{C})$ has basis

$$e = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \qquad f = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, \qquad h = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

with lie bracket

$$[h, e] = 2e, \qquad [h, f] = -2f, \qquad [e, f] = h.$$

This is the same for any k with $\operatorname{char} k \neq 2$.

1.1 "New" lie algebras from "old" ones

(1) Let $\mathfrak{g}$ be a lie algebra. One can get lie algebras with induced lie brackets as follows:

- **lie subalgebras** Consider a vector subspace $\mathfrak{h} \subseteq \mathfrak{g}$ such that $[h_1, h_2] \in \mathfrak{h}$ for all $h_1, h_2 \in \mathfrak{h}$.
- **ideal $I \lhd \mathfrak{g}$**. Let $I \subseteq \mathfrak{g}$ be a vector subspace such that $[x, y] \in I$ for $x \in I$ and $y \in \mathfrak{g}$.

Example 1.5. In the case of $\mathfrak{g} = \mathfrak{sl}_2(\mathbb{C})$, $\mathfrak{b} := \operatorname{span}\{e, h\} \subseteq \mathfrak{g}$ is a lie subalgebra, as well as $\mathfrak{b}' := \operatorname{span}\{f, e\} \subseteq \mathfrak{g}$.

However, these are not ideals. But $\operatorname{span}\{e\} \subseteq \mathfrak{b} = \operatorname{span}\{e, h\}$ is an ideal. Similarly, $\operatorname{span}\{f\} \subseteq \mathfrak{b}'$

If $I \lhd \mathfrak{g}$, then $\mathfrak{g}/I$ is again a lie algebra by setting

$$[x + I, y + I] := [x, y] + I.$$

You can check that this is well-defined since $I \lhd \mathfrak{g}$.

Remark 1.5.4. For a given $I \lhd \mathfrak{g}$, we have a short exact sequence of lie algebras

$$I \hookrightarrow \mathfrak{g} \xrightarrow{\text{can}} \mathfrak{g}/I$$

with $\operatorname{can}(x) = x + I$ being the canonical projection.

- For any lie algebra homomorphism $\varphi\colon \mathfrak{g}_1 \to \mathfrak{g}_2$, $\ker\varphi$ and $\operatorname{im}\varphi$ are lie (sub)algebras. You can check that e.g. if $x, y \in \ker\varphi$, then $\varphi([x, y]) = [\varphi(x), \varphi(y)] = [0, 0] = 0$, hence $[x, y] \in \ker\varphi$.

  In fact, $\ker\varphi \lhd \mathfrak{g}$ is an ideal.

- Consider the **center**

  $$Z(\mathfrak{g}) := \{x \in \mathfrak{g} \mid [x, y] = 0 \mid \forall y \in \mathfrak{g}\}.$$

  This is also a lie (sub)algebra. Here we check that if $z_1, z_2 \in Z(\mathfrak{g})$, we have

  $$\begin{aligned}[][[z_1, z_2], y] &= -[y, [z_1, z_2]] \\ &= [z_1, \underbrace{[z_2, y]}_{=0}] + [z_2, \underbrace{[y, z_1]}_{=0}].\end{aligned}$$

  In fact, $Z(\mathfrak{g}) \lhd \mathfrak{g}$.

- There is the **derived lie algebra** of $\mathfrak{g}$ given by

  $$\mathfrak{g}' := \operatorname{span}\{[x, y] \mid x, y \in \mathfrak{g}\}.$$

  Note that the span here is really necessary. For example, $\mathfrak{sl}_2(\mathbb{C})' = \mathfrak{sl}_2(\mathbb{C})$, but $\mathfrak{gl}_2(\mathbb{C})' \neq \mathfrak{gl}_2(\mathbb{C})$. In fact, $\mathfrak{gl}_2(\mathbb{C})' = \mathfrak{sl}_2(\mathbb{C})$.

- If $\mathfrak{g}_1, \mathfrak{g}_2$ are lie algebras, we can consider their **product**

  $$\mathfrak{g}_1 \times \mathfrak{g}_2 := \{(x, y) \in \mathfrak{g}_1 \oplus \mathfrak{g}_2 \mid x \in \mathfrak{g}_1, y \in \mathfrak{g}_2\},$$

  which becomes a lie algebra using the bracket

  $$[(x, y), (x', y')] := ([x, x'], [y, y']) \qquad \forall x, x' \in \mathfrak{g}_1, y, y' \in \mathfrak{g}_2$$

- Let $\mathfrak{g}$ be a lie algebra and R a commutative R-algebra. Then $\mathfrak{g} \otimes_k R$ is a lie algebra via

  $$[x \otimes r, x' \otimes r'] := [x, x'] \otimes rr' \ \forall x, x' \in \mathfrak{g}, r, r' \in R.$$

  This is called the **scalar extension** of $\mathfrak{g}$ by R.

Example 1.6 (special)**.** Take $R = k[t, t^{-1}]$ as the Laurent polynomials. This gives the **loop lie algebra** [1]

$$\mathcal{L}(\mathfrak{g}) := \mathfrak{g} \otimes_k k[t, t^{-1}].$$

More explicitly, $[x \otimes t^m, y \otimes t^n] = [x, y] \otimes t^{m+n}$.

This comes from looking at loops in $\mathfrak{g}$:

$$\left\{ f\colon S^1 \to \mathfrak{g} \;\middle|\; \text{f has a Fourier expansion}, f(t) = \sum_{i \in \mathbb{Z}} a_i t^i, \text{almost all } a_i = 0 \right\}$$

Example 1.7. As another special case, $R = k[t]$ is the **polynomial algebra**, and $\hat{\mathfrak{g}} := \mathfrak{g} \otimes_k k[t]$ is the **current algebra**.

(2) The **central extensions** of a lie algebra $\mathfrak{g}$. Let

$$\tilde{\mathfrak{g}} := \mathfrak{g} \oplus \underbrace{kc}_{\text{1-dimensional vector space with basis } c}.$$

Can we put a lie algebra structure on $\tilde{\mathfrak{g}}$ such that $\mathfrak{g} \hookrightarrow \tilde{\mathfrak{g}}$ is a lie subalgebra and $kc \in Z(\tilde{\mathfrak{g}})$?

We can define the lie bracket by

$$[x + \lambda c, y + \mu c] := [x, y] + \lambda\mu\Theta(x, y)c \qquad x, y \in \mathfrak{g}, \lambda, \mu \in k. \tag{$\star$}$$

Here, $\Theta\colon \mathfrak{g} \times \mathfrak{g} \to k = kc$ is a function satisfying

C1 Θ is bilinear.

C2 $\Theta(x, x) = 0$.

C3 $\Theta(x, [y, z]) + \Theta(y, [z, x]) + \Theta(z, [x, y]) = 0$.

Exercise. *Under the conditions (Cocyc0)-(Cocyc2), $\tilde{\mathfrak{g}}$ becomes a lie algebra with bracket ($\star$).*

Observe. *With this construction, hc is central in $\tilde{\mathfrak{g}}$, i.e. $hc \subseteq Z(\tilde{\mathfrak{g}})$. Also, we have a short exact sequence of lie algebras*

$$0 \longrightarrow kc \hookrightarrow \tilde{\mathfrak{g}} \twoheadrightarrow \mathfrak{g} \longrightarrow 0,$$

where kc is considered an abelian lie algebra.

Lecture 2
Fr 11 Apr 2025

[1] Often referred to just as the "loop algebra", even though this is incorrect, since it is not an algebra.

Orga 1.7.5. The first homework sheet is now online on Sciebo and due next Wednesday.

1.2 General framework: lie algebra extensions

Definition 1.8. An **extension** of a lie algebra $\mathfrak{g}$ by (a lie algebra) $\mathfrak{h}$ is a short exact sequence of lie algebras

$$0 \longrightarrow \mathfrak{h} \xhookrightarrow{i} \tilde{\mathfrak{g}} \xrightarrow{\pi} \mathfrak{g} \longrightarrow 0.$$

We call the extension **central** if $i(\mathfrak{h}) \subseteq Z(\tilde{\mathfrak{g}})$.

The extension is said to **split** if there exists a lie algebra homeomorphism $\beta\colon \mathfrak{g} \to \tilde{\mathfrak{g}}$ such that

Example 1.9. Consider $\tilde{\mathfrak{g}} = \mathfrak{h} \times \mathfrak{g}$ where i and π0are the obvious inclusions and projections, yielding the **trivial extension**

$$\mathfrak{h} \hookrightarrow \tilde{\mathfrak{g}} \longrightarrow \mathfrak{g}$$

$$h \longmapsto (h,0) \quad (h,x) \longmapsto x.$$

This splits with the lie algebra homeomorphism $\beta(x) = (0,x)$, by definition of $[-,-]$ on $\tilde{\mathfrak{g}}$. It is central if $\mathfrak{h}$ is abelian.

Warning. *In general, the morphism β need not exist.*

Proposition 1.10 (I.1)**.** 1) Given a central extension with a **linear** map $\beta\colon \mathfrak{g} \to \tilde{\mathfrak{g}}$ such that $\pi \circ \beta \ = \mathrm{id}$, then define

$$\Theta(x,y) := \Theta_\beta(x,y) := i^{-1}\left(\underbrace{[\beta(x),\beta(y)] - \beta([x,y])}_{\in \ker \pi = \operatorname{im} i}\right) : \mathfrak{g} \times \mathfrak{g} \to \mathfrak{h}.$$

Then

a) Θ is a 2-cocycle, i.e. it satisfies (Cocyc0)-(Cocyc2) with values in $\mathfrak{h}$.

b) There is a lie bracket on $\mathfrak{g} \oplus \mathfrak{h}$ defined as

$$[(x,h),(x',h')] := [x,x']_{\mathfrak{g}} + \Theta(x,x') \quad x,x' \in \mathfrak{g}\ h,h' \in \mathfrak{h} \qquad (\star)$$

such that

$$\Phi : \left| \begin{array}{ccc} \mathfrak{g} \oplus \mathfrak{h} & \longrightarrow & \tilde{\mathfrak{g}} \\ (x,h) & \longmapsto & \beta(x) + i(h) \end{array} \right.$$

is an isomorphism of lie algebras, where on the left hand side we take the lie bracket in ($\star$).

2) Given a 2-cocycle $\Theta\colon \mathfrak{g} \times \mathfrak{g} \to \mathfrak{h}$ (i.e. a map satisfying cocyc0-cocyc2) with values in an abelian lie algebra $\mathfrak{h}$.

Then $\mathfrak{g} \oplus \mathfrak{h}$ becomes a lie algebra denoted $\mathfrak{g} \ltimes_\Theta \mathfrak{h}$ via ($\star$). It defines a central extension

$$\mathfrak{h} \hookrightarrow \mathfrak{g} \ltimes_\Theta \mathfrak{h} \twoheadrightarrow \mathfrak{g}.$$

Proof. 1) a) Θ is clearly bilinear, since both β and the brackets are bilinear. $\Theta(x,x) = 0$ is clear since β, i are linear and $[-,-]$ is antisymmetric. The Jacobi identity for Θ is a calculation using the Jacobi identity of the lie bracket $[-,-]$.

b) We calculate that

$$[(x,h),(x,h)] = [x,x]_\mathfrak{g} + \Theta(x,x) = 0 + 0 = 0,$$

since $\mathfrak{g}$ is a lie algebra and Θ satisfies cocyc1. For the Jacobi identity, notice that the calculation just arises to the Jacobi identity on $\mathfrak{g}$ and Cocyc2 for Θ.

It remains to show that Φ is an isomorphism of lie algebras. Clearly, this is an isomorphism of vector spaces, so we only need to show that it is a lie algebra homeomorphism. We calculate

$$\begin{aligned}
\Phi([x,h],(x',h')] &= \beta([x,x']) + i(\Theta(x,x')) \\
\big[\Phi(x,h),\Phi(x',h')\big] &= [\beta(x) + i(h), \beta(x') + i(h')] \\
&= \underbrace{\big[\beta(x),\beta(x')\big]}_{=0} + \underbrace{\big[\beta(x),i(h')\big]}_{=0} + [i(h),\beta(x')] + \underbrace{\big[i(h),i(h')\big]}_{=0}
\end{aligned}$$

and equality follows..

2) We need to show that ($\star$) defines a lie bracket. This is just a calculation using $[-,-]$ on $\mathfrak{g}$ and 2-cocycle condition. It is clear that $\mathfrak{h} \hookrightarrow \mathfrak{g} \ltimes_\Theta \mathfrak{h} \twoheadrightarrow \mathfrak{g}$ is a short exact sequence of vector spaces, so it only remains to check whether i and π are lie algebra homeomorphisms. We compute

$$\begin{aligned}
\pi([x,h],[x',h']) = \pi([x,x'] + \Theta(x,x')) &= [x,x'] + 0 \\
\big[\pi(x,h),\pi(x',h')\big] &= [x,x'].
\end{aligned}$$

The extension is central, because

$$[(0,h'),(x,h)] \overset{(\star)}{=} [0,x] + \Theta(0,x) = 0$$

since $[-,-]$ on $\mathfrak{g}$ and Θ are both bilinear. □

Oral remark 1.10.6. This theorem gives rise to switching between extensions and 2-cocycles. We would hope for this to be a bijection, ideally, but this is not the case, since the construction depends on the choice of β.

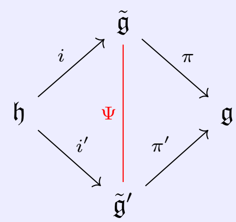
Definition 1.11 (Equivalence of extensions)**.** Given two extensions

$$\mathfrak{h} \xhookrightarrow{i} \tilde{\mathfrak{g}} \xrightarrow{\pi} \mathfrak{g}$$

and

$$\mathfrak{h}' \xhookrightarrow{i'} \tilde{\mathfrak{g}}' \xrightarrow{\pi'} \mathfrak{g}'$$

of $\mathfrak{g}$ by $\mathfrak{h}$, we say that these are **equivalent**, if there exists a lie algebra isomorphism Ψ such that the diagram



commutes.

Lemma 1.12 (I.2)**.** A central extension splits iff it is equivalent to a trivial extension.

Proof. Assume that $\mathfrak{h} \hookrightarrow \tilde{\mathfrak{g}} \twoheadrightarrow \mathfrak{g}$ splits, hence a lie algebra morphism $\beta \colon \mathfrak{g} \to \tilde{\mathfrak{g}}$ exists. We define

$$\Psi(h, x) := i(h) + \beta(x) \quad h \in \mathfrak{h}, x \in \mathfrak{g},$$

which is clearly an isomorphism of vector spaces, i.e. it only remains to check that it is a lie algebra homeomorphism:

$$\begin{aligned} \Psi([h, x), (h', x')]) &= \Psi([h, h'] + [x, x']) = i([h, h']) + \beta([x, x']) \\ \big[\Psi(h, x), \Psi(h', x')\big] &= \big[i(h) + \beta(x), i(h') + \beta(x')\big] \\ &= \big[i(h), i(h')\big] + \underbrace{\big[i(h'), \beta(x')\big]}_{=0} + \underbrace{\big[\beta(x), i(h')\big]}_{=0} + [\beta(x), \beta(x')]. \end{aligned}$$

This holds because i and β are lie algebra homeomorphisms and because the extension is central.

For the converse direction, if Ψ exists, then $\Psi(0, x) = \beta(x)$ gives a split map. $\square$

Lemma 1.13 (I.3)**.** A central extension $\mathfrak{g} \ltimes_\Theta \mathfrak{h}$ splits if and only if there exists a $\mu \in \mathrm{Hom}_k(\mathfrak{g}, \mathfrak{h})$ such that

$$\Theta(x,y) = \mu([x,y]) \quad \forall x, y \in \mathfrak{g}.$$

Proof. Assume that $\mathfrak{g} \ltimes_\Theta \mathfrak{h}$ splits, so there exists a split map $\beta\colon \mathfrak{g} \to \mathfrak{g} \oplus \mathfrak{h}$, say $\beta(x) = x + \mu(x)$ with $\mu(x) \in \mathfrak{h}$. Clearly, $\mu(x) \in \mathrm{Hom}_k(\mathfrak{g}, \mathfrak{h})$. Moreover,

$$\begin{aligned} [\beta(x), \beta(y)] &= [x + \mu(x), y + \mu(y)] \\ &\overset{(\star)}{=} [x,y] + \Theta(x,y) \end{aligned}$$

but also $\beta([x,y]) = \ldots$

Conversely, we have to check that Θ is a 2-cocycle. Clearly, it is bilinear and $\Theta(x,x) = \mu([x,x]) = \mu(0) = 0$. The (C2) condition follows from the Jacobi identity. Hence, $\mathfrak{g} \ltimes_\Theta \mathfrak{h}$0is defined.

Set $\beta(x) = x + \mu(x)$, where $\mu(x) \in \mathfrak{h}$ by assumption and hence this is split. Additionally,

$$\begin{aligned} [\beta(x), \beta(x')] &= [x + \mu(x), x' + \mu(x')] \\ &\overset{\text{lie bracket in } \mathfrak{g} \ltimes_\Theta \mathfrak{g}}{=} [x,x'] + \Theta(x,x') \\ \beta([x,x']) &= [x,x'] + \mu([x,x']) \\ &= [x,x'] + \Theta(x,x') \end{aligned}$$

□

Definition 1.14. Given lie algebras $\mathfrak{g}$ and $\mathfrak{h}$ with $\mathfrak{h}$ abelian, define vector spaces

$$\mathrm{Alt}^2(\mathfrak{g}, hfrka) = \{\Theta \text{ bilinear}, \Theta\colon \mathfrak{g} \times \mathfrak{g} \to \mathfrak{h} \mid \Theta(x,x) = 0 \ \forall x \in \mathfrak{g}\}.$$

$$Z^2(\mathfrak{g}, \mathfrak{h}) = \{\Theta\colon \mathfrak{g} \times \mathfrak{g} \to \mathfrak{h} \mid \Theta \text{ is a 2-cocycle}\}$$

$$B^2(\mathfrak{g}, \mathfrak{h}) := \{\Theta\colon \mathfrak{g} \times \mathfrak{g} \to \mathfrak{h} \mid \Theta \text{ is a 2-cocycle}, \exists \mu \in \mathrm{Hom}_k(\mathfrak{g}, \mathfrak{h} \text{ such that } \Theta(x,y) = \mu([x,y]) \ \forall x, y \in \mathfrak{g}\}.$$

Note $B^2 \subseteq Z^2 \subseteq \mathrm{Alt}^2$. Then, define

$$H^2(\mathfrak{g}, \mathfrak{h}) := Z^2(\mathfrak{g}, \mathfrak{h}) \big/ B^2(\mathfrak{g}, \mathfrak{h}$$

as the **second lie algebra cohomology of $\mathfrak{g}$ with values in $\mathfrak{h}$**.

Theorem 1.15 (I.4)**.** There is a bijection of sets

$$\begin{array}{ccl} H^2(\mathfrak{g},\mathfrak{h}) & \overset{1:1}{\longleftrightarrow} & \{\text{equivalence classes of central lie alg. extension of } \mathfrak{g} \text{ by } \mathfrak{h}\} \\ \overline{\Theta} & \longmapsto & \mathfrak{g} \ltimes_\Theta \mathfrak{h} \\ \overline{\Theta}_\beta & \longleftarrow & (\mathfrak{h} \hookrightarrow \tilde{\mathfrak{g}} \to \mathfrak{g}) \end{array}$$

Proof. We only need to show that the given maps are well-defined. The rest will follow from Proposition 1.10.

We will show the independence of the choice of β here. Assume that β, β' are linear split maps, then consider $\Theta_\beta - \Theta_{\beta'}$. We have

$$\begin{aligned} \Theta_\beta(x,y) - \Theta_{\beta'}(x,y) &= i^{-1}(\underbrace{[\beta(x),\beta(y)]}_{=0} - \beta([x,y])) - i^{-1}(\underbrace{[\beta'(x),\beta'(y)]}_{=0} + \beta'([x,y])) \\ &= (\beta' - \beta)([x,y]) \end{aligned}$$

Hence by setting $\mu := \beta' - \beta \in \mathrm{Hom}_k(\mathfrak{g},\mathfrak{h})$ we deduce $\Theta_\beta - \Theta_{\beta'} \in B^2(\mathfrak{g},\mathfrak{h})$ and therefore $\overline{\Theta_\beta} = \overline{\Theta_{\beta'}}$.

We will omit the proof that $\mathfrak{g} \ltimes_\Theta \mathfrak{h} \sim \mathfrak{g} \ltimes_{\Theta'} \mathfrak{h}$ if $\overline{\Theta} = \overline{\Theta'}$. □

Orga 1.15.7. Next time, we will look at examples of extensions. For now, let us state some definitions needed for the current exercise sheet.

Definition 1.16 (Representation of lie algebra)**.** A **representation** of a lie algebra $\mathfrak{g}$ is a k-vector space V together with a morphism of lie algebras $\mathfrak{g} \to \mathfrak{gl}(V)$.

Lemma 1.17 (I.5)**.** The data (V,φ) of a representation of a lie algebra $\mathfrak{g}$ is equivalent to a linear map

$$\hat{\varphi}\colon \mathfrak{g} \otimes V \to V, \quad \text{denoted } x.v := \hat{\varphi}(x \otimes v)$$

such that

$$[x,y].v = x.(y.v) - y.(x.v).$$

Proof. Let (V,φ) be a representation of $\mathfrak{g}$. Define $\hat{\varphi}(x \otimes v) := \varphi(x)(v)$. This satisfies

$$\begin{aligned} [x,y].v &= \varphi([x,y])(v) \\ &\overset{\varphi \text{ lie alg hom}}{=} [\varphi(x),\varphi(v)]v \\ &\overset{\text{lie bracket in } \mathfrak{gl}(V)}{=} (\varphi(x)\varphi(y) - \varphi(y)\varphi(x))(v) \\ &= \varphi(x)(\varphi(y)(v)) - \varphi(y)(\varphi(x)(v)) \\ &= x.(y.v) - y.(x.v) \end{aligned}$$

Orga 1.17.8. The other direction will be done next time.

□

Oral remark 1.17.9. This is a very important lemma. We will no longer distinguish between these two versions of stating a representation from now on.

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