Optimization and Game Theory in Flow Problems

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Summary of lectures

1 Introduction

Lecture 1
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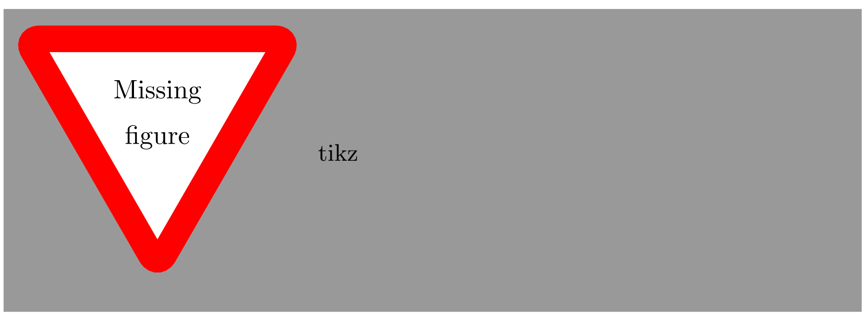
Question 1.0.1. Which problems can be modelled as flow problems?

Example 1.1. One interesting application are *Traffic Flows*.

- For example, a central authority might coordinate how people evacuate in case of a nearing flood. This is an example of a **flow over time**
- In a typical real-life situation, however, there is no such central authority that dictates peoples' routes. Thus, the setting resembles much rather a *game theoretical problem*, since each participant will individually decide which route to take, potentially leading to solutions that are not globally optimal.

  This can lead to unexpected outcomes:

Example 1.2 (Breass's Paradox)**.** Consider the following traffic network, where we want to route a total flow of 1:



Here, an edge of cost x will have cost equal to the amount of flow routed along it.

The optimum solution (in terms of maximum travel time) is of course to route flow $\frac{1}{2}$ along each of the paths.

It can be shown that this is also the solution that people will actually choose, since it is always better to use the currently less-congested path, leading to an equal spread.

However, now considering adding an additional .

2 Max Flows over time

Definition 2.1 (Flow over time)**.** Consider a graph $G = (V, E)$ with **transit times** $\{\tau_e\}_{e \in E}$ and capacities $\{u_e\}_{e \in E}$.

A **flow over time** f time horizon T is a Lebesgue integrable function $f_e \colon [0, T) \to \mathbb{R}_{\geqslant 0}$ for $e \in E$ such that $f_e(\theta) = 0$ for $\theta \geqslant T - \tau_e$.

Oral remark 2.1.2. The assumption on the Lebesgue integrability is just here for technical reasons, we will not deal much with it. In practice, most functions that we will encounter are piecewise constant.

Remark† **2.1.3.** We can interpret this definition as edges signaling pipes, and the flow f_e denoting the inflow rate to edge e over a certain time. Then, it takes τ_e time for the flow to traverse the pipes. Thus, $f_e(\theta - \tau_e)$ denotes the excess flow of edge e at time θ.

Definition 2.3. Let f be a flow over time with horizon T.

a) The **capacity constraint** is that $f_e(\theta) \leqslant u_e$ for all e, θ.

b) The **excess for a vertex** $v \in V$ at θ is defined as

$$\mathrm{ex}_f(v, \theta) \coloneqq \sum_{e \in \delta^-(v)} \int_0^{\zeta - \tau_e} f_e(\zeta) d\zeta - \sum_{e \in \delta^+(v)} \int_0^{\zeta} f_e(\zeta) d\zeta$$

c) We say that f has **weak flow conservation** if

$$\mathrm{ex}_f(v, \theta) \quad \geqslant \quad 0 \forall v \in V \setminus \{s\} \, \forall \theta \in [0, T)$$
$$\mathrm{ex}_f(v, T) = 0 \forall v \in V \setminus \{s, t\}$$

d) We say that f has **strict flow conservation** if

$$\mathrm{ex}_f(v, \theta) = 0 \forall v \in V \, \{s, t\} \, \forall \theta \in T$$

e) The **value of f** is $|f| \coloneqq \mathrm{ex}_f(t, T)$.

f) A flow is called **feasible** if it satisfies the capacity constraints and has weak flow conservation.

Consider a graph $G = (V, E)$ with capacities $c \colon E \to \mathbb{R}_{\geqslant 0}$, transit times $\tau \colon E \to \mathbb{R}_{\geqslant 0}$, a time horizon T and source/sink vertices s, t.

The Maximum Flow over Time Problem asks for a feasible s-t-flow over time with time horizon T and maximum value of $|f|$.

Definition 2.4 (Temporally repeated flow with time horizon T)**.** Let x be a static [1]s-t (feasible) flow and consider a flow decomposition $(X_p)_{p\in\mathcal{P}\cup\mathcal{C}}$ with $|\mathcal{P}\cup\mathcal{C}| \leqslant |E|$.

Now, define a flow over time by

$$f_e(\theta) := \sum_{P\in P_e(\theta)} x_p \qquad \forall e=(v,w)\in E\ \theta\in[0,T),$$

where

$$P_e(\theta) := \left\{p\in\mathcal{P} \;\middle|\; p\in\mathcal{P}, \tau(P_{s,v})\leqslant\theta, \tau(P_{v,t})\leqslant T-\theta\right\}.$$

Here, the transit times along a fixed path P are defined canonically as

$$\tau(P_{v,w}) := \sum_{e\in P_{v,w}} \tau_e.$$

The resulting flow is called the **temporally repeated flow with time horizon T** associated to the static flow x.

Observation 2.5. *The Temporally repeated flow can be obtained as follows: For each $P\in\mathcal{P}$, we send x_p into P from s during $[0,T-\tau(P))$ and then pass the flow along P.*

Note that the temporally repeated flow even fulfills strict flow conservation.

Lemma 2.7. Let x be a static, feasible s-t flow with flow decomposition $(x_P)_{P\in\mathcal{P}\cup\mathcal{C}}$ and $x_P = 0$ for all $P\in\mathcal{P}$ with $\tau(P) > T$ and for all cycles. Denote by f the corresponding temporally repeated flow f.

Then,

$$|f| = T\cdot|x| - \sum_{e\in E}\tau_e\cdot x_e.$$

Proof. Motivated by Observation 2.5, we deduce:

$$\begin{aligned}
|f| &= \sum_{P\in\mathcal{P}} (T-\tau(P))\cdot x_P \\
&= T\cdot\sum_{P\in\mathcal{P}} x_P - \sum_{P\in\mathcal{P}}\tau(P)\cdot x_P \\
&= T\cdot|x| - \sum_{e\in E}\tau_e \underbrace{\sum_{\substack{P\in\mathcal{P}\\ e\in P}} x_p}_{=x_e}
\end{aligned}$$

□

[1]Here, "static" means just "not temporal"

Corollary 2.8. Let x be a static, feasible s-t flow and $(x_P)_{P\in\mathcal{P}\cup\mathcal{C}}$ a flow decomposition. Then,

$$|f| \geqslant T\cdot|x| - \sum_{e\in E}\tau_e x_e.$$

Proof. Define a new flow by

$$\tilde{x}_P := \begin{cases} x_P & \text{if } P\in\mathcal{P} \text{ and } \tau(P)\leqslant T \\ 0 & \text{otherwise} \end{cases}$$

for all $P\in\mathcal{P}\cup\mathcal{C}$. Then, the associated temporally repeated flows satisfy

$$\begin{aligned} |f| &= \left|\tilde{f}\right| \\ &= T|\tilde{x}| - \sum_{e\in E}\tau_e\tilde{x}_e \\ &\geqslant T|x| - \sum_{e\in E}\tau_e x_e \end{aligned}$$

Note that to check the last inequality, it suffices to see that cycles only contribute to the negative sum and do not change the flow value of $\tilde{x}$ compared to x. For paths P with $\tau(P) > T$, these contribute to $|x|$ but not to $\tilde{x}$, however, the effect cancels out by summing the τ_e along this path in the negative summand. □

Algorithmus 1 : Ford-Furkerson Algorithm for maximum flows over time

Input : $G=(V,E)$ with capacities c, transit times τ, time horizon T and source/sink vertices s, t.

Output : A feasible temporally repeated flow

Compute a feasible static s-t-flow x maximizing

$$T\cdot|x| - \sum_{e\in E}\tau_e x_e$$

Compute a flow decomposition $(x_P)_{P\in\mathcal{P}\cup\mathcal{C}}$.
Output corresponding temporally repeated flow.

Remark 2.8.4. To implement step 1, consider the auxiliary graph G' obtained from G by adding an edge (t,s) with infinite capacity. For each edge, set its cost to τ_e.

Now, for each static s-t flow x in G gives rise to a circulation in G' by sending $|x|$ along (t, s), whose cost will be precisely

$$\sum_{e \in E} x_e \cdot \tau_e + |x| \cdot (-T).$$

Thus, solving the minimum cost flow problem in G' yields a desired flow f.